Programming Paradigms: Exercises

March 3, 2016

1 Abstract machines

These questions are about the von Neumann-machine and the interpreter for the functional language from the first lecture. To solve these exercises, edit the files and load them into the Haskell REPL (Read-Eval Print Loop) `ghci`.

1.1 Virtual Imperative Machine

Exercise 1 Make sure you can run the examples given in the file `VM.hs`. [*]

Exercise 2 What does the example loop program below do? [*]

```
exLoop = Machine
         [Load 0 "zero"
         ,Load 1 "one"
         ,Load 1000 "counter"
         ,Print "counter"
         ,BinOp Sub "counter" "one" "counter"
         ,BinOp Neq "counter" "zero" "cond"
         ,RelJump "cond" (-4)
         ,Halt]
         initMemory
```

Express it in some language with imperative features of your choice (e.g. C(++), Java(Script), Python, ...).

Answer to exercise 2 A faithful rendering of the program:

```
int zero = 0;
int one  = 1;
int counter = 1000;
loop:
printf("%d\n",counter);
```

```
counter = counter - one;
int cond = counter != zero;
if (cond) goto loop;
```

Using more idiomatic C:

```
int counter = 1000;
while (counter != 0) {
  printf("%d\n",counter);
  counter--;
}
```

Exercise 3 How do you multiply two numbers without using `Mul`? [*]

Answer to exercise 3 By using `Add` and a jump.

Exercise 4 How do you formulate an if-statement in the language? Translate this snippet that prints the maximum of two numbers. Run it with the memory initialized for the variables `x`, `y`, etc: [*]

```
if (x > y) {
  print(x);
} else {
  print(y);
}
```

Exercise 5 How do you calculate the absolute value of a number? [*]

Answer to exercise 5 Check if the value is greater than zero. If it isn't, subtract it from zero.

Exercise 6 Make a function that loops a program forever. It takes a program as input and it returns another program that runs the input program forever. [**]

An appropriate type of the loop is:

```
loop :: [Instruction] -> [Instruction]
```

The following program should print `1` indefinitely:

```
exOnes = Machine (loop [Print "one"]) [("ipLoc",0),("one",1)]
```

(press Ctrl+C in `ghci` to stop it!)

The following Haskell functions are helpful:

```
(++)   :: [a] -> [a] -> [a] -- xs ++ ys concatenates the two lists xs and ys
length :: [a] -> Int        -- length of a list (to calculate the jump offset)
```

Use your loop constructor to print the infinite sequence of triangle numbers 0, 1, 3, 6, 10, ... (calculated as 0, 0+1, 0+1+2, 0+1+2+3, ...).

Answer to exercise 6

```
loop :: [Instruction] -> [Instruction]
loop instr =
  instr ++
  [ Load 1 "_always"
  , RelJump "_always" (- (length instr + 2))
  ]

exTriangle = Machine
    ([ Load 0 "counter"
     , Load 0 "inc"
     , Load 1 "one"
     ] ++
     loop
       [ Print "counter"
       , BinOp Add "inc" "one" "inc"
       , BinOp Add "counter" "inc" "counter"
       ])
    initMemory
```

Exercise 7 Make a program builder function for if statements with greater-than. This amounts to making a function with the following type: [**]

```
ifGt :: Address -> Address -> [Instruction] -> [Instruction]
                                            -> [Instruction]
```

Thus a call to `ifeq addr1 addr2 ins1 ins2` checks if the value in `addr1` is greater than the one in `addr2`. If it is, it runs the instructions `ins1`, otherwise `ins2`.

Test your implementation by making a program that prints the maximum of three numbers.

Answer to exercise 7

```
-- A slightly more generic version
ifThenElse :: Op -> Address -> Address ->
  [Instruction] -> [Instruction] -> [Instruction]
ifThenElse op addr1 addr2 true_ins false_ins =
  [ BinOp op addr1 addr2 "_cond"
  , RelJump "_cond" (length false_ins + 2)
  ]
  ++ false_ins ++
  [ Load 1 "_always"
  , RelJump "_always" (length true_ins) ]
  ++ true_ins

ifGt :: Address -> Address ->
  [Instruction] -> [Instruction] -> [Instruction]
ifGt = ifThenElse Gt

exGreatest x y z = Machine
  (ifGt "x" "y"
    (ifGt "x" "z"
      [Print "x"]
      (ifGt "y" "z" [Print "y"] [Print "z"]))
    (ifGt "y" "z" [Print "y"] [Print "z"])
   ++ [Halt])
  [(ipLoc,0), ("x",x), ("y",y), ("z",z)]
```

1.2 Functional Language

Exercise 8 A) What `Value` does `eval (Lam "x" (Var "x"))` evaluate to? [*]
You cannot print this value in `ghci`, but which value is it?

Answer to exercise 8

```
  eval (Lam "x" (Var "x"))
= Fun (\ v -> eval (subst "x" v (Var "x")))
= Fun (\ v -> eval (Val v))
= Fun (\ v -> v)
```

B) What happens, step by step, in the evaluation of

```
eval (App (Lam "x" (Var "x")) (Val (Number 1)))?
```

Answer to exercise 8

```
  eval (App (Lam "x" (Var "x")) (Val (Number 1)))
= let Fun f' = eval (App (Lam "x" (Var "x")))
                             in f' (eval (Val (Number 1)))
= let Fun f' = Fun (\ x -> x) in f' (eval (Val (Number 1)))
= let Fun f' = Fun (\ x -> x) in f' (Number 1)
= (\ x -> x) (Number 1)
= Number 1
```

Exercise 9 Make sure you can run the examples given in the file `Fun.hs`. [*]

Exercise 10 Import the `max` function from Haskell using `Bin`, and use it to write a lambda term that calculates the maximum of three numbers. [*]

Answer to exercise 10

```
max' = Bin max
max3 = Lam "x" $ Lam "y" $ Lam "z" $
         max' (Var "x") (max' (Var "y") (Var "z"))
```

Exercise 11 Add an if-then-else construct to the language where false is represented with the number zero. [**]

Add the constructor `IfThenElse Expr Expr Expr` to the `Expr`-datatype, with the evaluation semantics that `IfThenElse p t f` means the expression `p` is evaluated, and if it is nonzero evaluates `t`, otherwise `f`.

Import the `>` function (greater-than) from Haskell using `Bin` and the interpretation of true and false as above, and use it and your `IfThenElse`-construct to write a lambda term that calculates the maximum of three numbers.

Answer to exercise 11

```
eval e0 = case e0 of
  IfThenElse p t f ->
    case eval p of
```

```
      Number 0 -> eval f
      _        -> eval t
  ...

subst what for e0 = case e0 of
  IfThenElse p t f ->
    IfThenElse (subst what for p) (subst what for t)
                                  (subst what for f)
  ...

gt = Bin (\ x y -> if x > y then 1 else 0)

max3' = Lam "x" $ Lam "y" $ Lam "z" $
  IfThenElse (gt (Var "x") (Var "y"))
    (IfThenElse (gt (Var "x") (Var "z"))
      (Var "x")
      (Var "z"))
    (IfThenElse (gt (Var "y") (Var "z"))
      (Var "y")
      (Var "z"))
```

Exercise 12 Show that the `Bin` constructor is redundant by making an equivalent function that does to use it: [***]

```
bin :: Op -> Expr -> Expr -> Expr
```

Hint: Use `Val`.

Answer to exercise 12

```
bin :: Op -> Expr -> Expr -> Expr
bin op e1 e2
  = Val (Fun (\ (Number x) ->
          Fun (\ (Number y) ->
            Number (x `op` y))))
  @@ e1 @@ e2
```

2 Generalities

2.1 Paradigms, Languages, Features

Consider the language C++ (or your favourite programming language, ...).

Exercise 13 Write a list of features (programming constructs) implemented in this language. Be as exhaustive as you can (list at least 10 features). [*]

Exercise 14 For each programming paradigm (Imperative, OO, etc.), evaluate how well this language supports that paradigm. Argue using the list compiled in the previous answer. [*]

Exercise 15 Can you identify a paradigm not studied in the course which is supported by this language? [***]

2.2 Types

Exercise 16 Give a meaningful type to the following values. [*]

1. 4
2. 123.53
3. 1/3
4. π
5. 'c'
6. "Hello, world!"
7. -3
8. (unary) -
9. (binary) +
10. sin
11. derivative

Exercise 17 Explain the meaning of the following types. (Hint: what kind of values can inhabit those types?) [**]

1. String
2. String $\rightarrow$ String
3. String $\rightarrow$ String $\rightarrow$ String
4. (String $\rightarrow$ String) $\rightarrow$ String
5. String $\rightarrow$ (String $\rightarrow$ String)
6. (String $\rightarrow$ String) $\rightarrow$ (String $\rightarrow$ String)

Answer to exercise 17 Example of values which have these types:

1. a string constant
2. a function that reverses/doubles/etc a string
3. a concatenation function
4. a function that evaluates a function on some constant string
5. same as 3
6. a function that reverses a function

One can not only parameterize over values, but also over types. (Eg. in Java, generic classes).

For example, the following type is a suitable type for a sorting function: it expresses that the function works for any element type, as long as you can compare its inhabitants.

$\forall$ a. (a $\rightarrow$ a $\rightarrow$ Bool) $\rightarrow$ Array a $\rightarrow$ Array a

Exercise 18 Does `sort` in your favourite language have a similar type? How close/far is it? [**]

Exercise 19 Consider the type [***]

$$\forall a\, b.\mathsf{Pair}\, a\, b \rightarrow \mathsf{Pair}\, b\, a$$

What programs can possibly have this type?

3 Imperative Programming

3.1 Gotos to loops

Consider the algorithm:

- **Preliminary:** A is an array of integers of length m and B is an array of integers of length n. Also the elements from both arrays are distinct (from the elements in both arrays) and in ascending order.
- **Step1:** if n or m is zero **STOP**. Otherwise if $m>n$, set $t := \lfloor log\,(m/n) \rfloor$ and go to **Step4**, else set $t := \lfloor log\,(n/m) \rfloor$.
- **Step2:** compare $A[m]$ with $B[n + 1 - 2^t]$. If $A[m]$ is smaller, set $n := n - 2^t$ and return to **Step1**.
- **Step3:** using binary search(which requires exactly t more comparisons), insert $A[m]$ into its proper place among $B[n + 1 - 2^t]$ ... $B[n]$. If k is maximal such that $B[k] < A[m]$, set $m := m - 1$ and $n := k$. Return to **Step1**.
- **Step4:**(Step 4 and 5 are like 2 and 4, interchanging the roles of n and m, A and B) if $B[n] < A[m+1-2^t]$, set $m := m - 2^t$ and return to **Step1**.
- **Step5:** insert $B[n]$ into its proper place among the As. If k is maximal such that $A[k] < B[n]$, set $m := k$ and $n := n - 1$. Return to **Step1**.

Exercise 20 Implement binary search without gotos in the context of the algorithm. There is a slight change compared to the classical algorithm, since the scope of the search is different. [1] **[*]**

Exercise 21 **Step1** may require the calculation of the expression $\lfloor log\,(m/n) \rfloor$, for $n \geq m$. Explain how to compute this easily without division or calculation of a logarithm. **[-]**

Exercise 22 Why does the binary search mentioned in the algorithm always take t steps ? **[*]**

Exercise 23 Explain the behaviour of the algorithm for the arrays $A = \{87, 503, 512\}$ and $B = \{61, 154, 170, 275, 426, 509, 612, 653, 677, 703, 765, 897, 908\}$. **[*]**

Exercise 24 Implement the above-mentioned algorithm without using *gotos* in a programming language of your choice. Check your implementation with the A and B from the previous question. **[**,@2]**

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[1] If that is too easy, do it for red-black trees.

Answer to exercise 24

```
void merge(int a[], int b[], int m, int n) {
  int a_size=m;
  int b_size=n;
  while (n != 0 && m != 0) {
    printf("%d %d\n",m,n);
    if (!(m > n)) {
      int t = log_2(n / m);
      int i = n + 1 - pow_2(t);
      if (a[m-1] < b[i-1]) {
        printf("Decreasing n\n");
        n = n - pow_2(t);
      } else {
        int k = binsearch(i-1,n,a[m-1],b)+1;
        printf("Inserting %d into b at %d\n", a[m-1], k-1);
        insert(a[m-1],k-1,b_size,b);
        b_size++;
        m = m - 1;
        n = k;
      }
    } else /* m > n */ {
      int t = log_2(m / n);
      int i = m + 1 - pow_2(t);
      if (b[n-1] < a[i-1]) {
        printf("Decreasing m\n");
        m = m - pow_2(t);
      } else {
        int k = binsearch(i-1,m,b[n-1],a)+1;
        printf("Inserting %d into a at %d\n", b[n-1], k-1);
        insert(b[n-1],k-1,a_size,a);
        a_size++;
        n = n - 1;
        m = k;
      }
    }
  }
  printf("%d %d\n",m,n);
}
```

3.2 Control flow statements to gotos

Exercise 25 Translate the following **for** loop with explicit gotos: [*,@2]

```
for (statement1; condition; statement2)
  loop_body
```

Answer to exercise 25

```
    statement1;
    loop:
    if (condition) {
        loop_body;
        statement2;
        goto loop;
    }
```

Exercise 26 Translate the following **foreach** loop with explicit gotos: [*]

```
  foreach i in k..l do
    body
```

Exercise 27 Translate the do/while construct. [*]

Exercise 28 Translate the switch/case construct. [*,@2]

(If you want to make sure your translation is correct you should check the specification of the C language. `http://www.open-std.org/jtc1/sc22/wg14/www/docs/n1124.pdf`)

Answer to exercise 28 The following `switch`-snippet

```
switch(a) {
    case pattern1: statement1; break;
    case pattern2: statement2;
    default: def_statement;
}
```

can be translated like this:

```
if (var == pattern1) goto case_1;
if (var == pattern2) goto case_2;
goto case_default;
```

```
case_1:       statement1; goto case_end;
case_2:       statement2;
case_default: def_statement;
case_end:
```

Note The following translation is incorrect due to the fall-through behaviour of switch in C.

```
if (var == pattern1) {
    statement1;
} else if (var == pattern2) {
    statement2;
} else
    def_statement;
```

Exercise 29 Translate the insertion sort algorithm. [*]

3.3 Pointers and call by reference

Exercise 30 Create a binary search tree where nodes contain integer number in C/C++/Java. Implement a function that adds a value to the tree and one that deletes a certain value from the tree. [*]

Exercise 31 Write a recursive pre-order traversal of the above tree. [*]

Answer to exercise 31

```
void preorder(Tree *t) {
    if (t != NULL) {
        printf("%d\n",t->v);
        dfs(t->l);
        dfs(t->r);
    }
}
```

Exercise 32 Write a swap function using call by reference. [*]

Answer to exercise 32 In C++ syntax:

```
void swap(int& a, int& b) {
    int t = a;
    a = b;
    b = a;
}
```

Exercise 33 Write a swap function using call by value. [*]

Answer to exercise 33 Merely using call by value won't work because only copies of values will be changed, local variables are destroyed when function returns. Note the absence of &'s in the signature.

```
void swap(int a, int b) {
    int t = a;
    a = b;
    b = t;
}
```

The standard way is then to pass pointers (make the references explicit):

```
void swap(int* a, int* b) {
    int t = *a;
    *a = *b;
    *b = t;
}
```

But it is also possible to use a copy parameters back and forth, for example using a structure.

```
typedef struct {
    int fst;
    int snd;
} pair;

struct pair swap(struct pair p) {
    int t = p.fst;
    p.fst = p.snd;
    p.snd = t;
    return p;
}
```

Exercise 34 Does Java use call by reference or call by value? Give examples to support your answer. [**]

Answer to exercise 34 Java calling convention is a mishmash:

- Scalars are passed by value
- Objects are passed by reference, but with a catch, see below.

```
void f(Object x) {
  x = null; -- this will only change the local copy of x
}
...
y = new Object();
f(y);

-- pitfall: y != null
```

In sum, the calling convention of Java is "what you would do normally in C"... Not a very clean semantics.

Exercise 35 Write down pros and cons of using call by reference vs. call by value. (Ease of use, performance, ...) [**]

Answer to exercise 35 Use call-by-value to protect the data from occasional corruption by a buggy procedure. Use call-by-reference if arguments are too big to be passed on stack.

3.4 More on control flow and pointers: Duff's device

Duff's device is an optimization idiom for serial copies in the language C. The fragment below lists first the original code, then Duff's optimization.

```c
/* (Almost) original code */
int main() {
   short *to, *from;
   int count;
   ...
   {
     /* pre: count > 0 */
     do
          *to++ = *from++;
     while(--count>0);
   }
   return 0;
}
```

Many things happen in the assignment “*to++ = *from++;”. Can you figure out what exactly?

Exercise 36 Translate the above to multiple statements, so that for each of them only one variable (or memory location) is updated. Explain in your own words what happens (draw a picture if necessary). [*]

> **Answer to exercise 36** It is equivalent to `*to = *from; to++; from++;`. In other words, the memory pointed by `from` is copied to the memory pointed by `to`, then the pointers are incremented.

Duff optimised the above code as follows:

```c
/* Duff's transformation */
int main() {
  short *to, *from;
  int count;
  ...
  {
     /* pre: count > 0 */
     int n = (count + 7) / 8;
     switch(count % 8){
       case 0:       do{        *to++ = *from++;
       case 7:                *to++ = *from++;
       case 6:                *to++ = *from++;
       case 5:                *to++ = *from++;
       case 4:                *to++ = *from++;
       case 3:                *to++ = *from++;
       case 2:                *to++ = *from++;
```

```
        case 1:                 *to++ = *from++;
                  } while (--n > 0);
      }
   }
   return 0;
}
```

Exercise 37 Translate the switch statements to gotos. [**]

Is the second program really equivalent to the first?

Exercise 38 Show that the instruction "`*to++ = *from++`" will be executed `count` times in the second program. [-]

> **Answer to exercise 38** The while loop will be executed $\lfloor (count+7)/8 \rfloor$ times. Since each iteration executes the instruction 8 times, we expect the program to run it $8 \times \lfloor (count+7)/8 \rfloor$ times in total, which is up to 7 times more than count.
>
> However, the `switch` instruction is there to skip *count* mod 8 instructions in *the first iteration*, to make the count exact.

Exercise 39 Explain the equivalence by a series of program transformations. [****]

Exercise 40 Can you guess why Duff expects the second program to be faster? What instructions are executed in the first program but not in the second? [-]

> **Answer to exercise 40** The test and jump instructions are executed only $\lfloor (count+7)/8 \rfloor$ times instead of *count*, which is about 1/8 of the time.

Further reading

- For the original email in which Tom Duff presented his "device", see `http://www.lysator.liu.se/c/duffs-device.html`
- You can see the assembly code generated by gcc by compiling with

  gcc -S <filename>.

3.5 From recursion to explicit stack

Exercise 41 Consider the following tree data structure: **[**,@2]**

```
struct tree {
    int v;
    struct tree *l, *r;
};

typedef struct tree Tree;
```

Implement a pre-order traversal, using explicit stack(s).

Answer to exercise 41 In the following, the variable s refers to the global stack.

```
void preorder_stack(Tree *u) {
    push(u);
    do {
        Tree *t = pop();
        if (t != NULL) {
            printf("%d\n",t->v);
            push(t->r);
            push(t->l);
        }
    } while (s);
}
```

Consider the following recursive equation for computing Fibonacci numbers:

$$fib_{n+2} = fib_{n+1} + fib_n$$

Exercise 42 Implement the recursive function computing the n-th Fibonacci number based on the expression above. [*]

Answer to exercise 42

```
int fib(int n) {
    int tmp,res;
    if (n < 1) {
        res = 0;
    } else if (n == 1) {
```

```
        res = 1;
    } else {
        int tmp = fib(n-1);
        res = tmp + fib(n-2);
    }
    return res;
}
```

Exercise 43 What is the complexity of the computation? (How many calls will there be to compute fib_n?) [*]

Answer to exercise 43 There will be approximately ϕ^n calls, where $\phi = \frac{1+\sqrt{5}}{2}$. ($\phi$ is the golden ratio.)

Exercise 44 Implement a slightly-optimized recursive function for the same purpose, using an accumulator parameter. [*]

Exercise 45 Implement a version of each of the two functions above by using an explicit stack. [**]

Answer to exercise 45 The non-optimised version yields:

```
int fib_stk(int arg) {
  int tmp, loc, result;
  // push the initial stack frame. Global caller has code 0
  push(arg,UNDEFINED,0);
 call:
  if (s->n < 1) {
    result = 0;
  } else if (s->n == 1) {
    result = 1;
  } else {
    push(s->n-1,UNDEFINED,1); // tmp not initialised here
    goto call;
  loc_1:
    tmp = result;
    push(s->n-2,tmp,2);
    goto call;
```

```
  loc_2:
    result = tmp + result;
  }
  // return:
  tmp = s->tmp; // restore local vars
  loc = s->loc; // remember the caller
  pop();
  // jump to the caller
  if (loc == 0) return result; // if global, return.
  if (loc == 1) goto loc_1;
  if (loc == 2) goto loc_2;
}
```

Consider the following function:

```
void move_many(int n, int source, int intermediate, int target) {
  if (n != 0) {
    move_many(n-1,source,target,intermediate);
    move_disk(source,target);
    move_many(n-1,intermediate,source,target);
  }
}
```

Exercise 46 Remove recursion, using an explicit stack. [**]

Exercise 47 Eliminate the tail call. (Do not use the stack for it). [**]

Exercise 48 The structure of the other call is particular: it is possible to recover the arguments of the caller from the arguments of the call. Use this property to avoid using the stack for these arguments. [***]

Exercise 49 Consider the Ackerman function: [**,@3]

```
int a(int m, int n) {
  if (m == 0)
    return n+1;
  else if (n == 0)
    return a(m-1,1);
  else {
    return a(m-1,a(m,n-1)); // note the non-tail call
  }
}
```

Transform the tail calls to loops.

Answer to exercise 49

```
int a(int m, int n) {
  while (m != 0) {
    if (n == 0) {
      m = m-1;
      n = 1;
    } else {
      m = m-1;
      n = a(m,n-1);
    }
  }
  return n+1;
}
```

Exercise 50 Implement the Ackermann function without recursion. (use a stack) [**]

Exercise 51 Implement the algorithm from the previous section without loops (only recursion allowed). [**]

Exercise 52 Translate the quicksort algorithm. [**]

Exercise 53 For each of the following: implement the algorithm as a recursive function. (And remove the loop!) [**]

1. ```
   -- n given
   x = 1
   while n > 0 do
     b = least_significant_bit(n);
     n = n / 2;
     x = x * x + b;
   return x
   ```

2. ```
   a = 0
   b = 1
   for i in [1..n] do
     c = a + b
     a = b
     b = c
   return a
   ```

4 Object-Oriented Programming

Consider the following code, in C# syntax:

```
interface Monoid {
   Monoid op(Monoid second);
   Monoid id();
};

struct IntegerAdditiveMonoid : Monoid {
    public IntegerAdditiveMonoid(int x) {
         elt = x;
    }
    public IntegerAdditiveMonoid op(IntegerAdditiveMonoid second) {
         return new IntegerAdditiveMonoid(
             elt + second.elt);
    }
    public IntegerAdditiveMonoid id(){
         return new IntegerAdditiveMonoid(0);
    }
    int elt;
};
```

4.1 Explicit method pointers

Exercise 54 Translate the above code to a C-like language, using explicit method pointers. Store the method pointers directly in the object. (Hint: you can simply consider the interface as a class without fields.) [*]

Exercise 55 Same as above, but using method tables. [*]

Answer to exercise 55

```
    #include "stdio.h"
    #include "stdlib.h"

struct monoid {
  struct monoid_vtable *vtable;
};

typedef struct monoid Monoid;

struct monoid_vtable {
  Monoid *(*op)(Monoid *this, Monoid *second);
```

```
  Monoid *(*id)(Monoid *this);
};

typedef struct monoid_vtable Monoid_Vtable;

struct integerAdditiveMonoid {
     Monoid* super;
     int elt;
};

typedef struct integerAdditiveMonoid IntegerAdditiveMonoid;

IntegerAdditiveMonoid *integerAdditiveMonoid_new(int x);

Monoid *integerAdditiveMonoid_op(Monoid *this, Monoid *second) {
     return (Monoid *)
         (integerAdditiveMonoid_new(((IntegerAdditiveMonoid*) this)->elt
         + ((IntegerAdditiveMonoid *) second)->elt));
};

Monoid *integerAdditiveMonoid_id(Monoid *this) {
     return (Monoid *) (integerAdditiveMonoid_new(0));
}

Monoid_Vtable integerAdditiveMonoid_vtable =
{  &integerAdditiveMonoid_op
  , &integerAdditiveMonoid_id };

IntegerAdditiveMonoid *integerAdditiveMonoid_new(int x) {
     IntegerAdditiveMonoid *this =
         malloc(sizeof(IntegerAdditiveMonoid));
     this->super = malloc(sizeof(Monoid));
     this->super->vtable = &integerAdditiveMonoid_vtable;
     this->elt = x;
     return this;
}

int main() {
     IntegerAdditiveMonoid *a = integerAdditiveMonoid_new(3);
     IntegerAdditiveMonoid *b = integerAdditiveMonoid_new(4);
     printf("%d\n",((IntegerAdditiveMonoid*)
         (a->super->vtable->op((Monoid *) a, (Monoid *) b)))->elt);
     printf("%d\n",((IntegerAdditiveMonoid*)
         (b->super->vtable->id((Monoid *) b)))->elt);
```

```
    free(a);
    free(b);
    return 0;
}
```

Exercise 56 Briefly recap: what is a *monoid*, mathematically? [-]

Exercise 57 Give two examples of data types that can be considered monoids. (Hint: Strings would form a monoid under the appropriate structure; what is the structure? [-]

Exercise 58 Write another instance of the monoid interface, using one of the examples you found. Also write its translation to a C-like language. [*]

Exercise 59 Assume variables a,b of type Monoid. Translate the expression a.op(b). [*]

Exercise 60 Assume to objects x,y of two different instances of Monoid are bound to the variables a,b. Explain what happens at runtime when the expression is evaluated. (Which code is executed?) [*]

4.2 Co/Contra variance

Surprise: the above C# code defining the IntegerAdditiveMonoid is refused by the C# compiler:

```
> gmcs Monoids.cs
Monoids.cs(6,8): error CS0535: ‘IntegerAdditiveMonoid’ does not implement
  interface member ‘Monoid.op(Monoid)’
Monoids.cs(2,11): (Location of the symbol related to previous error)
Monoids.cs(6,8): error CS0535: ‘IntegerAdditiveMonoid’ does not implement
  interface member ‘Monoid.id()’
Monoids.cs(3,11): (Location of the symbol related to previous error)
Compilation failed: 2 error(s), 0 warnings
```

Exercise 61 What if the method op *would* compile? Define objects a,b, of appropriate types, so that a.op(b), if is run, would result in a run-time error. [*]

Answer to exercise 61

```
    a = new IntegerAdditiveMonoid(3);
    b = new IntegerAdditiveMonoid(4);
    a.op(b);

error: method op() not found
```

Exercise 62 What if the method id would compile? Could you construct a similar run-time error? (Hint: do the translation job if the answer is not obvious to you.) [*]

Answer to exercise 62

```
    a = new IntegerAdditiveMonoid(3);
    a.id();

error: method id() not found
```

Exercise 63 Explain the error messages in terms of co-/contra-/nonvariance. [**]

Answer to exercise 63 In an interface implementation, types should be covariant, because the users of the interface expect to supply arguments of generic types and to get a return value of a generic type, not of the narrow ones.

Exercise 64 The corresponding program in Java behaves differently. Briefly explain how, consult the Java Language Specification (JLS), and back your answer with the appropriate clause(s) in the JLS. [***]

Exercise 65 Can you change the code so that the (current) C#-compiler accepts it? What is the problem then? [-]

5 Functional Programming

5.1 Algebraic Data Types and Pattern Matching

Exercise 66 Write functions between the types [*]

$$(Either\,A\,B, Either\,C\,D)$$

and

$$Either (Either (A, C)\,(B, D))\,(Either (A, D)\,(B, C))$$

.

Answer to exercise 66

```
f :: (Either a b,Either c d) ->
     Either (Either (a,c) (b,d)) (Either (a,d) (b,c))
f (Left a,Left c)   = Left (Left (a,c))
f (Left a,Right d)  = Right (Left (a,d))
f (Right b,Left c)  = Right (Right (b,c))
f (Right b,Right d) = Left (Right (b,d))

g :: Either (Either (a,c) (b,d)) (Either (a,d) (b,c)) ->
     (Either a b,Either c d)
g (Left (Left (a,c)))   = (Left a,Left c)
g (Left (Right (b,d)))  = (Right b,Right d)
g (Right (Left (a,d)))  = (Left a,Right d)
g (Right (Right (b,c))) = (Right b,Left c)
```

Exercise 67 Define an algebraic type for binary trees [*]

Answer to exercise 67

```
data Tree a = Branch (Tree a) a (Tree a) | Empty
```

Exercise 68 define an algebraic type for arithmetic expressions (+, *, ...) **[*,@3]**

Answer to exercise 68

```
data Expr
    = Lit Int
    | Add Expr Expr
    | Mul Expr Expr
    | Var String
```

Exercise 69 Define a simple interpreter for the above type; that is given an expression, compute its value. **[*,@3]**

Answer to exercise 69

```
eval :: [(String,Int)] -> Expr -> Expr
eval env (Lit x)   = x
eval env (Add u v) = eval env u + eval env v
eval env (Mul u v) = eval env u * eval env v
eval env (Var v)   = case lookup v env of
    Just x  -> x
    Nothing -> error $ "Unbound variable " ++ v
```

Exercise 70 Translate the above 2 structures to an OO language. (Hint: One class corresponds to leaves, one to branching.) **[*,@3]**

Answer to exercise 70

```
interface Tree<A> {}

class Branch<A> implements Tree<A> {
    private Tree<A> left,right;
    private A element;

    public Branch (Tree<A> l, A e, Tree<A> r) {
        left = l;
        element = e;
        right = r;
    }
}

class Empty<A> implements Tree<A> {
    public Empty () {}
}
```

Exercise 71 Translate the interpreter to an OO language. You are not allowed to use 'instanceof' **[*,@3]**

Answer to exercise 71

```
import java.util.HashMap;

interface Expr {
    public Integer eval(HashMap<String,Integer> env);
}

class Lit implements Expr {
    private Integer x;

    public Lit(int x) {
        this.x = x;
    }

    public Integer eval(HashMap<String,Integer> env) {
        return x;
    }

    public Integer val() {
        return x;
    }
}

class Add implements Expr {
    private Expr l,r;

    public Add(Expr l,Expr r) {
        this.l = l;
        this.r = r;
    }

    public Integer eval(HashMap<String,Integer> env) {
        return l.eval(env) + r.eval(env);
    }

    public Expr left() {
        return l;
    }

    public Expr right() {
        return r;
    }
}
```

```
class Mul implements Expr {
    private Expr l,r;

    public Mul(Expr l,Expr r) {
        this.l = l;
        this.r = r;
    }

    public Integer eval(HashMap<String,Integer> env) {
        return l.eval(env) * r.eval(env);
    }

    public Expr left() {
        return l;
    }

    public Expr right() {
        return r;
    }
}

class Var implements Expr {
    private String v;

    public Var(String v) {
        this.v = v;
    }

    public Integer eval(HashMap<String,Integer> env) {
        return env.get(v);
    }

    public String var() {
        return v;
    }
}
```

Exercise 72 Translate the interpreter to an OO language. You must use 'instanceof'. [*]

Answer to exercise 72

```java
import java.util.HashMap;

class Eval
{
    public Integer eval(HashMap<String,Integer> env, Expr e) {
        if (e instanceof Lit) {
            Lit u = (Lit)e;
            return u.val();
        } else if (e instanceof Add) {
            Add u = (Add)e;
            return eval(env,u.left()) + eval(env,u.right());
        } else if (e instanceof Mul) {
            Mul u = (Mul)e;
            return eval(env,u.left()) * eval(env,u.right());
        } else if (e instanceof Var) {
            Var u = (Var)e;
            return env.get(u.var());
        } else {
            return null;
        }
    }
}
```

5.2 Currification and partial application

Exercise 73 Define a function `f` following this spec: given a integer, return it unchanged if it is greater than zero, and zero otherwise. (The type must be `Int` $\rightarrow$ `Int`.) [*]

Answer to exercise 73

```haskell
f :: Int -> Int
f x | x > 0     = x
    | otherwise = 0
```

Exercise 74 Assuming a function max : `(Int` $\times$ `Int)` $\rightarrow$ `Int`, define the function `f`. [*]

Answer to exercise 74

```
f' :: Int -> Int
f' x = max (x,0)
```

Exercise 75 Define a function `max'`, by currifying the function `max`. [*]

Answer to exercise 75

```
max' :: Int -> Int -> Int
max' x y = max (x,y)
```

Exercise 76 Define `f` using `max'`. [*]

Answer to exercise 76

```
f'' :: Int -> Int
f'' = max' 0
```

5.3 Higher-order functions

Assume the `filter`, `map`, `foldr` functions as in the Haskell prelude. `f` comes from the previous section.

Exercise 77 For each of the following expressions, remove occurences of the higher-order functions by inlining them. [*]

```
exp1 = map f
exp2 = filter (>= 0)
exp3 = foldr (:) []
exp4 = foldr (++) []
exp5 = foldr (\ x xs -> xs ++ [x]) []
```

Answer to exercise 77

```
exp1 = map f
  = go where
      go [] = []
      go (x:xs)
        | x > 0     = x : go xs
        | otherwise = 0 : go xs

exp2 = filter (>= 0)
  = go where
      go [] = []
      go (x:xs) | x >= 0    = x : go xs
                | otherwise = go xs

-- Recap: definition of foldr
foldr f z []     = z
foldr f z (x:xs) = f x (foldr f z (x:xs))

exp3 = foldr (:) []
  = (go where
      go [] = []
      go (x:xs) = x : go xs)
  (= id :: [a] -> [a])

exp4 = foldr (++) []
  = (go where
      go [] = []
      go (x:xs) = x ++ go xs)
  (= concat :: [[a]] -> [a])

exp5 = foldr (\ x xs -> xs ++ [x]) []
  = (go where
      go [] = []
      go (x:xs) = (\ x xs -> xs ++ [x]) x (go xs))
  = (go' where
      go' [] = []
      go' (x:xs) = go' xs ++ [x])
  (= reverse :: [a] -> [a])
```

Consider the following imperative program:

```
for (i=0;i<sizeof(a);i++)
  if (a[i].grade >= 24)
    *b++ = a[i];
```

Exercise 78 How would the same algorithm be naturally expressed in a functional language? (Use functions from the Haskell prelude to shorten your code) [*,@3]

Answer to exercise 78

```
f      = filter ((>= 24) . grade)

f' xs = [ x | x <- xs, grade x >= 24 ]
```

Assume we represent a vector as a list of integers.

Exercise 79 write a function that does the dot-product of two vectors using explicit recursion and pattern matching. [*]

Answer to exercise 79 This can be done with two functions, or with an accumulating parameter. The latter is shown here:

```
dotacc acc []     []     = acc
dotacc acc (x:xs) (y:ys) = dotacc (acc + x*y) xs ys
dotacc acc _      _      = error $ "unequal lengths!"

dot = dotacc 0
```

Exercise 80 In the above code, abstract over sum and products on integers. [*]

Answer to exercise 80 We use `sum` and `zipWith`, from `Data.List` (answers the next question, too). Note that this solution does not error if the lists are of different lengths (unlike the answer to the previous question).

```
dot' xs ys = sum (zipWith (*) xs ys)
```

Exercise 81 Can you find the functions you created in the Haskell `Data.List` module? [**]

Exercise 82 The standard function `insert` inserts an element into an ordered list, for example []

```
insert 4 [1,3,5,7] = [1,3,4,5,7]
```

Define a function

```
sort :: [Integer] -> [Integer]
```

to sort lists, using `insert`.

Answer to exercise 82

```
sort :: [Integer] -> [Integer]
sort []     = []
sort (x:xs) = insert x (sort xs)
```

Exercise 83 Express `sort` in terms of `foldr` (do not use explicit recursion). []

Answer to exercise 83

```
sort' :: [Integer] -> [Integer]
sort' = foldr insert []
```

5.4 Closures

Consider the program:

```
test1 xs = foldr (+) 0 xs
test2 xs = foldr (*) 1 xs
```

Exercise 84 Identify higher-order applications in the above program. [*]

Answer to exercise 84 `foldr`, `(+)` and `(*)`.

Exercise 85 Assuming that `xs` are lists of integers, replace the above uses of higher-order application by explicit closures. (Hint: you need to also change the definition of `foldr`). [*]

Answer to exercise 85

```
data Closure = Add | Mul

applyCl :: Closure -> Integer -> Integer -> Integer
applyCl Add x y = x + y
applyCl Mul x y = x * y

foldr :: Closure -> Integer -> [Integer] -> Integer
foldr cl z []     = z
foldr cl z (x:xs) = applyCl cl x (foldr cl z xs)

test1 xs = foldr Add 0 xs   -- triangular numbers
test2 xs = foldr Mul 1 xs   -- factorials
```

Exercise 86 Add the following lines to the above program and repeat the above 2 exercises. [**]

```
replace a b xs = map (\x -> if x == a then b else x) xs
moment n xs = sum (map (\x -> x ^ n) xs)
```

Answer to exercise 86

```
data Closure
    = ReplaceLambda Integer Integer
    -- ^ a and b from @\ x -> if x == a then b else x@
    | MomentLambda Integer
    -- ^ n from @\ x -> x ^ n@

applyCl :: Closure -> Integer -> Integer
applyCl (ReplaceLambda a b) x = if x == a then b else x
applyCl (MomentLambda n) x = x ^ n

map :: Closure -> [Integer] -> [Integer]
map c []     = []
```

```
map c (x:xs) = applyCl c x : map c xs

replace a b xs = map (ReplaceLambda a b) xs
moment n xs = sum (map (MomentLambda n) xs)
```

Exercise 87 Eratosthenes' sieve is a method for finding prime numbers, dating from ancient Greece. We start by writing down all the numbers from 2 up to some limit, such as 1000. Then we repeatedly do the following: [*]

- The *first* number in the list is a prime. We generate it as an output.
- We *remove all multiples* of the first number from the list—including that number itself.
- Loop.

We terminate when no numbers remain in our list. At this point, all prime numbers up to the limit have been found.

Write the algorithm in Haskell.

Answer to exercise 87 See next question.

Exercise 88 Consider the following algorithm: [*]

```
primes = sieve [2..1000]
sieve (n:ns) = n:sieve (filter (not . (`isDivisibleBy` n)) ns)
x `isDivisibleBy` y = x `mod` y == 0
```

Inline the call of function composition in the above as a lambda abstraction. The only remaining higher-order function is filter.

Exercise 89 Write a version of filter which takes an explicit closure as an argument. Remember to write (and use) an apply function. Hint: in order to test your program you need to define an example parameter, such as (>= 0), etc. [*]

Exercise 90 Re-write sieve using the above. [*]

Exercise 91 Write a version of sieve in imperative (or OO) language, by translation of the answer to the previous exercise. (eg. you can not use arrays instead of lists). (Hint: write this one as a recursive program). [**]

5.5 Explicit state

Consider a binary tree structure with integer values in the nodes.

Exercise 92 In the Java version, write a function that replaces each element with its index in preorder traversal of the tree. [*]

Exercise 93 Translate the function above to Haskell thinking of it as an imperative algorithm. Use `State` from the *ContinueState* lecture. What is the "state of the world" in this case? [**]

Exercise 94 Rewrite the Haskell version, in such a way that the continuations are made visible, i.e. eliminate your usage of `State`. [*]

5.6 Laziness

There are no lazy languages that permit mutation.

Exercise 95 Why not? [*** ,@4]

5.6.1 Lazy Dynamic Programming

Consider a the function computing the fibonacci sequence:

```
fib 0 = 0
fib 1 = 1
fib n = fib (n-1) + fib (n-2)
```

Exercise 96 Estimate the time-complexity of computing $\mathsf{fib}\, n$. [-]

One can make the above computation take linear time by saving the intermediate results in an array:

```
fibs[0] = 0
fibs[1] = 1
for i = 2 to n do
  fibs[n] = fibs[n-1] + fibs[n-2]
```

Exercise 97 Instead of specifying the order of filling the array via an imperative algorithm, let Haskell's lazy evaluation take care of it. Write the definition of the array fibs in Haskell. [*]

Exercise 98 What portion of the array is forced when $\mathsf{fibs}!k$ is accessed? Draw the graph of computation dependencies for $k = 5$. [*]

Exercise 99 Write the Floyd-Warshall algorithm in your favourite imperative language. Assume that there is no weight on edges; and you're only interested in whether there is a path between two given nodes, not the number of steps in the path. [*]

Note that the best-case performance is Cubic.

Exercise 100 Repeat the above, but make sure you never overwrite a cell in a matrix. (What do you need to do to make this at all possible?) [*]

Exercise 101 Using the same formula, fill the Floyd-Warshall matrix using an array comprehension in a lazy language (optionally use explicit thunks for the contents of each cell). Discuss the best-case performance. [*]

Exercise 102 Does your favourite spreadsheet program have strict, or lazy logical disjunction operator? Test it. [-]

Exercise 103 Can you write the Floyd-Warshall algorithm in it? [-]

Exercise 104 Repeat the above steps with the algorithm to compute the least edit distance. `http://en.wikipedia.org/wiki/Levenshtein_distance` **[**,@4]**

```
int LevenshteinDistance(char s[1..m], char t[1..n])
{
  // for all i and j, d[i,j] will hold the Levenshtein distance between
  // the first i characters of s and the first j characters of t;
  // note that d has (m+1)x(n+1) values
  declare int d[0..m, 0..n]

  for i from 0 to m
    d[i, 0] := i // the distance of any first string to an empty second string
  for j from 0 to n
    d[0, j] := j // the distance of any second string to an empty first string

  for j from 1 to n
  {
    for i from 1 to m
    {
      if s[i] = t[j] then
        d[i, j] := d[i-1, j-1]       // no operation required
      else
        d[i, j] := minimum
                   (
                     d[i-1, j] + 1,  // a deletion
                     d[i, j-1] + 1,  // an insertion
                     d[i-1, j-1] + 1 // a substitution
```

```
                      )
    }
  }

  return d[m,n]
}
```

5.6.2 Lazy Lists (Streams)

Remember the sieve algorithm. Lazy lists can be (potentially) infinite, although of course no program can evaluate *all* the elements of an infinite list. Nevertheless, using lazy lists can help us avoid building unnecessary limitations into our code. Check that the `sieve` function also works given the *infinite* list `[2..]`—the output should be the *infinite* list of all prime numbers.

Exercise 105 How would you find the first 100 prime numbers? **[*]**

Hint: start by answering the following questions on a simpler algorithm, for example the function `enumFrom` which generates the infinite list of numbers starting at a given one.

Exercise 106 Translate `sieve` to use explicit thunks. Oh noes, this introduced higher-order function(s). **[*,@4]**

Exercise 107 Where are the new higher-order functions? **[**]**

You know what's coming...

Exercise 108 Remove higher-orderness using the best technique available. **[***]**

Exercise 109 Write a version of lazy sieve in imperative (or OO) language. **[**,@4]**

Since there are no more functions in the data, you can now display your infinite lists.

Exercise 110 Do so for a few meaningful inputs, and interpret what you see. **[***]**

6 Concurrent Programming

6.1 Channels and Processes

Exercise 111 Write a process that manages a bank account. It has a channel for queries; which can be deposits and withdrawals. The answer is posted to a **[*,@6]**

channel that comes with the query. (Note: you cannot use references to store any state – see variable-managing process for inspiration)

Exercise 112 Write a client for the above process that move "money" from an account to another. ("transaction") **[*,@6]**

Exercise 113 Assume that withdrawals/deposits can fail. (For example if there is too much/litte on the account). Modify the server process accordingly. **[*,@6]**

Exercise 114 Is the total amount of money a constant of your system? (Consider that transactions may fail in the "middle".) How can you ensure that it is? Write the corresponding code. **[**,@6]**

Exercise 115 Implement a process that manages a semaphore. The process should implement the requests P and V. (See the wikipedia article for explanation of semaphore, P and V). `http://en.wikipedia.org/wiki/Semaphore_(programming)` [*]

Exercise 116 Implement two library functions that help communicate with the above server. (Hint: you may have to create channels.) [*]

6.2 Explicit continuations

Consider the following outline for the "business logic" of a web-application:

```
session connection = do
   items <- webForm connection "What do you want to buy?"
   address <- webForm connection "Where do you want your stuff delivered?"
   daMoney <- webForm connection "Enter your credit card details, now."
   secureInDatabase daMoney (priceOf items)
   placeOrder items address
```

Exercise 117 What is the purpose, and type of the webForm primitive? **[*,@7]**

Exercise 118 Transform the type of webForm to take a continuation. **[**,@7]**

Exercise 119 Break the above code into continuations, linked by the webForm primitive. [**,@7]

Exercise 120 Outline what the webForm function should do. Discuss in particular what happens if the user presses the "back" button in their browser. [***,@7]

Recursion and continuations Remember your interpreter for arithmetic expressions. It should have type:

Expr → Int

Let's make continuations explicit. In this case, the result is not returned directly, but applied to a continuation. Hence, the type becomes:

Expr → (Int → a) → a

Exercise 121 Write a wrapper for the interpreter which has the above type [*]

Exercise 122 Replace every recursive call in the interpreter by a call to the wrapper. (Hint: you must decide what is the "continuation" for every call, and for this you must choose an order of evaluation!) [***]

Exercise 123 Unfold the usage of the interpreter in the wrapper. [***]

7 Logic Programming

In this section, exercises are sometimes formulated both in Prolog and Curry syntax; as indicated in the margin.

7.1 Metavariables and unification

Exercise 124 What is the result of each of the following unifications? Try to come up with the result without using the prolog/curry interpreter! [*,@7]

Prolog

```
a(X,Y) = a(b,c)
a(X,Y) = a(Z,Z)
a(X,X) = a(b,c)
e(X) = a(b,b)
d(X,X,Y) = d(Z,W,W)
a(X,X) = d(Z,W,W)
```

Curry

```
data X = A X X | B | C | D X X X | E X

A x y =:= A B C  where x, y free
A x y =:= A z z  where x, y, z free
A x x =:= A B C where x free
E x =:= A B B where x free
D x x y =:= D z w w where x, y, w, z free
A x x =:= D z w w where x, y, w, z free
```

Exercise 125 Assume a relation plus relating two natural numbers and their sum. [*]

Define a relation minus, relating two natural numbers and their difference, in terms of plus.

7.1.1 Difference Lists

A difference list is a special structure that can support efficient concatenation. It uses unification in a clever way to that end.

The difference-list representation for a list can be obtained as follows:

Prolog

```
fromList([],d(X,X)).
fromList([A|As],d(A:Out,In)) :- fromList(As,d(Out,In)).
```

Curry

```
data DList a = D [a] [a]

fromList :: [a] -> D a -> Success
fromList [] (D x x') = x =:= x'
fromList (a:as) (D (a':o) i) = a =:= a' & fromList as (D o i)
```

A structure of the form d(Out,In) will represent the list L if Out unifies with L concatenated with In. Or, less technically, a list L will be represented as the difference between Out and In: so for instance,

Prolog

$$[1,2] \longrightarrow d([1,2,3,4],[3,4])$$

Curry

$$[1,2] \longrightarrow D[1,2,3,4][3,4]$$

You can check how fromList works by testing it: Prolog

```
fromList([1,2,3],X).
```

Curry

```
fromList [1,2,3] x where x free
```

Note that the same metavariable (G271 in my implementation) is present twice in the result. Note that we can get the original result back by unifying this metavariable with the empty list.

Exercise 126 Write a predicate toList :: DList a → [a] → Success to get back to the normal list representation. [**]

Given that representation for lists, it is possible to perform concatenation by doing unification only!

Exercise 127 Write a predicate dconcat to concatenate two difference lists, without using the direct representation. [**,@7]

If `dconcat` is properly defined, then the queries Prolog

```
dconcat(X,Y,Z), fromList([1,2,3],X), fromList([4,5],Y), toList(Z, Final).
```

Curry

```
dconcat x y z & fromList [1,2,3] x & fromList [4,5] y & toList z final
   where x, y, z, final free
```

should produce:

```
...
final = [1,2,3,4,5]
```

Exercise 128 What happens when you concatenate a difference list with itself? [***]

7.2 Functions ↔ Relations

Consider the following haskell function, that splits a list of integers into two lists: one containing the positive ones (and zero), the other containing the negative ones.

```
split [] = ([],[])
split (x:xs) | x >= 0 = (x:p,n)
             | x <  0 = (p,x:n)
  where (p,n) = split xs
```

If written as a predicate, it is natural if it takes 3 arguments. For example,

```
split([3,4,-5,-1,0,4,-9],p,n)
```

should bind:

```
P = [3,4,0,4]
N = [-5,-1,-9].
```

Exercise 129 Write the predicate split. Make sure that it is written in relational style. That is, do not use any function (only other relations and constructors). [**]

Exercise 130 What are the lists that are returned by split when used in reverse? Can it fail? [**]

7.3 Explicit Search

Consider the following relation.

```
ancestor x y = parent x y
ancestor x y = ancestor x z & parent z y
  where z free
```

Exercise 131 Convert the program from relational to functional style. That is, write a function `ancestors` that returns all the ancestors of a given person. More precisely, given a person `c`, construct the list of `x`s such that `ancestor x c` is true. [*,@8]

Consider the following list comprehension, in Haskell syntax:

```
c = [f x | z <- a, y <- g z, x <- h y, p v]
```

Exercise 132 Write down possible types for f, a, g, h and p. [*]

Exercise 133 Assume that the above functions/values (f, a, g, h and p) are translated to relational style. What would be natural types for them? [*]

Exercise 134 Translate the list comprehension to relational style. [*]

Exercise 135 Translate all the functions from Family.curry in the "list of successes" style, for all directions. [**]